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Spectral asymmetry and supersymmetry

Frank Ferrari^{a,b}^a *Institut de Physique, Université de Neuchâtel rue A.-L. Bréguet 1, CH-2000 Neuchâtel, Switzerland*^b *Centre National de la Recherche Scientifique, Laboratoire de Physique Théorique de l'École Normale Supérieure,
24 rue Lhomond, 75231 Paris Cedex 05, France*

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Abstract

Fractional charges, and in particular the spectral asymmetry η of certain Dirac operators, can appear in the central charge of supersymmetric field theories. This yields unexpected analyticity constraints on η from which classic results can be recovered in an elegant way. The method could also be applied in the context of string theory.

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1. Introduction

A field theory with fundamental fields carrying integer charges only can have sectors in the Hilbert space with fractionally charged states. Such states obviously cannot be created by any finite action of the local fundamental fields. Their existence can be inferred in the context of a semi-classical analysis [1], where they correspond to solitons, particle-like solutions of the classical field equations. This important phenomenon has many applications, in particular for polymers [2] and the quantum Hall effect [3]. I recommend the excellent recent review by Wilczek [4] for more details. The purpose of the present note is to point out some interesting properties of the fractional charges, that have not been discussed previously in spite of the long history of the subject. We will describe in particular an elegant way to recover the classic results. We are also

able to give some exact formulas for the charges in some specific models, that go beyond the usual semi-classical approximation. The ideas we will discuss can be extended from the usual field theory setting to the more general string theory setting, where the study of charge fractionization is still in its infancy.

A common example of a charge that can be fractionated is the fermion number F . The Dirac hamiltonian in a soliton background has in general a non-trivial energy spectrum with a density of eigenvalues $\rho(E)$. Semi-classically, the fermion number can be computed in a standard way by expanding the Dirac spinor ψ in terms of positive and negative energy eigenstates. This yields

$$F = \frac{1}{2} \int dx \langle [\psi^\dagger, \psi] \rangle = -\frac{\eta}{2}, \quad (1)$$

where η is the so-called spectral asymmetry [5], the difference between the number of positive and

E-mail address: frank.ferrari@unine.ch (F. Ferrari).

negative energy eigenstates,

$$\eta = \lim_{h \rightarrow 0} \int dE \rho(E) \text{sign}(E) |\lambda|^{-h}. \quad (2)$$

The formulas (2) and (1) have, respectively, a mod 2 and a mod 1 ambiguity when the Dirac operator has zero modes. This is due to degenerate lowest soliton states with different fermion numbers. For example, when a conjugation symmetry relates states with opposite energies, only the zero modes can contribute to the fermion number. With k complex zero modes, F can then take any of the $k + 1$ different values $-k/2, -k/2 + 1, \dots, +k/2$. In the most interesting and generic cases, there is no zero mode and no conjugation symmetry, and all the eigenvalues can contribute to F . As shown in [6], the fermion number is then in general irrational. A detailed analysis of this problem, with many applications and references, can be found in [7].

A typical example is the fractional fermion number of a magnetic monopole in an SU(2) four-dimensional Yang–Mills theory with an adjoint Higgs field. The Dirac equation is

$$\gamma^\mu (\partial_\mu + i A_\mu) \psi = (m_1 + \phi_1 - i \gamma^5 (m_2 + \phi_2)) \psi, \quad (3)$$

where m_1 and m_2 are real mass parameters, ϕ_1 and ϕ_2 are real adjoint Higgs fields, and ψ is a Dirac spinor of charge $F = 1$. Asymptotically, the background fields ϕ_j tend to the Higgs vacuum expectation values $\langle \phi_j \rangle = a_j \sigma^3$. The integer magnetic number p is given by the magnetic flux through the sphere at infinity,

$$\frac{1}{8\pi a} \int_{S_\infty} dS^i \epsilon_{ijk} \text{tr}(\phi F^{jk}) = p, \quad \text{where } a = a_1 + i a_2 \quad \text{and} \quad \phi = \phi_1 + i \phi_2. \quad (4)$$

The fermion number for this problem has been calculated in the literature in cases of increasing generality. In the conjugation symmetric case $m_2 = a_2 = 0$, the number of zero modes $k = p$ can be derived using Callias' index theorem [8]. When $m_1 = a_2 = 0$, the formula for F was given in [6], and when $a_2 = 0$ it was given in [9],

$$F = \frac{p}{2\pi} \left[\arctan\left(\frac{m_1 - a_1}{m_2}\right) - \arctan\left(\frac{m_1 + a_1}{m_2}\right) \right]. \quad (5)$$

We will consider the more general Dirac operator for which m_1 , m_2 , a_1 and a_2 can all be non-zero because it is the case that naturally arises in our approach.

The formula (5) has a curious property that has not been discussed before:

The fermion number or, equivalently, the spectral asymmetry is a harmonic function of the parameters.

The simplest proof of this statement is given by noting that F is the imaginary part of a holomorphic function. By introducing the complex parameters $m = m_1 + i m_2$ and $a = a_1 + i a_2$, we have indeed

$$F = \frac{p}{2\pi} \text{Im} \ln \frac{m + a}{m - a}. \quad (6)$$

The logarithm is defined with the branch cut on the negative real axis and the argument of a complex number between $-\pi$ and π . When $a_2 = 0$ we then recover (5), and we will prove in the next section that (6) is the correct generalization. The real part of the holomorphic function contains the terms $\ln|m \pm a|$, which are the logarithms of the eigenvalues of the fermion mass matrix in (3). This suggests the more precise statement:

The fermion number or equivalently the spectral asymmetry is given by the imaginary part of a holomorphic function whose real part can be deduced from the one-loop low-energy effective coupling of some field theory containing the fermion ψ .

This result is powerful, because it relates a rather involved calculation of the fermion number in a solitonic sector to a trivial one-loop calculation in the vacuum sector. Moreover, the validity of this result is not limited to the four-dimensional Dirac operator in the monopole background. For example, in the two-dimensional version of (3), the background fields ϕ_j describe a kink solution, $\lim_{x \rightarrow \pm\infty} \phi_j = \phi_{j,\pm}$, and the vector potential A_μ , that goes to a pure gauge at infinity, implement a possible gauge symmetry. The fermion number has been calculated in [6,7,10,11] for $A_\mu = 0$. By introducing $\phi_\pm = \phi_{1,\pm} + i \phi_{2,\pm}$ it can be put in the form

$$F = \frac{1}{2\pi} \text{Im} \ln \frac{m + \phi_-}{m + \phi_+}, \quad (7)$$

which has the same qualitative features as (6).

2. Charge fractionization and supersymmetry

The properties of η discussed above can be checked on the final formulas, but are rather strange and unexpected from the point of view of the standard approach to the problem [7]. We will now present a framework that makes those properties very natural, and from which formulas like (6) or (7) are easily derived. The idea is to embed the problem in a supersymmetric setting. Of course supersymmetry is not fundamental in our problem, since the objects that we consider—the spectral asymmetry of a Dirac operator or more generally fractional charges—are defined and mostly used in a non-supersymmetric context. But the point is that supersymmetry is a nice mathematical tool that provides an interesting new point of view on those objects. The fact that the phenomenon of charge fractionization can play an important role in the physics of supersymmetric field theories was emphasized in [12]. In some sense, we will show that the arguments of [12] can be used backwards to infer results on charge fractionization.

The coupling of a Dirac fermion to a vector potential and a complex adjoint Higgs field as described by (3) occurs in $\mathcal{N} = 2$ supersymmetric Yang–Mills theories in four dimensions with one flavor of quark of complex bare mass m . The Dirac fermion belongs to the quark hypermultiplet, and the coupling to A_μ and ϕ is determined by gauge invariance and supersymmetry. The formula of fundamental importance to us is a certain anticommutator of the supersymmetry charges,

$$\{Q_\alpha^I, Q_\beta^J\} = 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{IJ}Z, \quad (8)$$

where α and β are spinorial indices and I and J , $1 \leq I, J \leq 2$, label the supersymmetry charges. The bosonic charge Z is called the central charge of the supersymmetry algebra. Classically, it is a linear combination of the electric charge Q_e , the magnetic charge Q_m and the fermion number F ,

$$Z_{\text{cl}} = \frac{2a(Q_e + iQ_m)}{g_{\text{YM}}} + mF, \quad (9)$$

where g_{YM} is the Yang–Mills coupling constant. For us, the most important property of Z , valid in the full quantum theory, is the following:

The central charge Z is a holomorphic function of a and m , such that

$$\frac{\partial Z}{\partial a} = p\tau_{\text{eff}} + q, \quad (10)$$

where p and q are the integer-valued magnetic and electric quantum numbers, respectively, and τ_{eff} is the complexified low-energy effective coupling constant defined in terms of the effective Yang–Mills coupling and effective topological theta angle as

$$\tau_{\text{eff}} = \frac{\theta_{\text{eff}}}{\pi} + \frac{8i\pi}{g_{\text{YM,eff}}^2}. \quad (11)$$

This result comes from the fact that Z can be calculated from the low-energy effective action [13], which is governed by a single holomorphic function \mathcal{F} called the prepotential such that $\partial^2 \mathcal{F} / \partial a^2 = \tau_{\text{eff}}$ [14]. The analyticity property can also be deduced from supersymmetric Ward identities.

The fact that the real charge F contributes to the holomorphic function Z gives a natural explanation of the harmonicity properties discussed in section one. To make this idea quantitative, we need a formula expressing F in terms of Z in the full quantum theory. This is a priori non-trivial, because a derivation of the quantum version of (9) from the low-energy effective action has not appear when $m \neq 0$. The result, however, is suggested by the quantum analysis of the electric charge. The Witten effect [15] in the low energy theory implies that

$$\frac{2Q_e}{g_{\text{YM,eff}}} = q + \frac{p\theta_{\text{eff}}}{\pi} = \text{Re} \frac{\partial Z}{\partial a}. \quad (12)$$

Now, (3) shows that the Higgs field couple to the electric charge in the same way as m couple to the fermion number. We thus propose that the correct quantum formula for the fermion number is simply

$$F = \text{Re} \frac{\partial Z}{\partial m}. \quad (13)$$

Eqs. (12) and (13) show that both $Q_e/g_{\text{YM,eff}}$ and F are harmonic functions of the parameters. Semi-classically, the electric charge is related to a quantity, similar to the spectral asymmetry, involving the Dirac operator (3). The methods used to calculate η can be straightforwardly adapted for $a_2 = 0$ [16], and indeed yield a harmonic function,

$$\frac{2Q_e}{g_{\text{YM,eff}}} = q - \frac{p}{2\pi} \left[\arctan\left(\frac{m_1 - a_1}{m_2}\right) + \arctan\left(\frac{m_1 + a_1}{m_2}\right) \right]. \quad (14)$$

We still have to understand the relation to a one-loop effective coupling constant. The idea is that the real parts of the derivatives of Z , which are difficult to obtain directly, can be deduced from the imaginary parts by using holomorphy. The imaginary parts turn out to be particularly easy to calculate. The Dirac quantization condition [17] implies that

$$\frac{2Q_m}{g_{\text{YM,eff}}} = \frac{8\pi p}{g_{\text{YM,eff}}^2} = \text{Im} \frac{\partial Z}{\partial a}. \quad (15)$$

A standard non-renormalization theorem [18] states that $g_{\text{YM,eff}}^2$ is given to all orders of perturbation theory by one-loop Feynman diagrams, and that there is also a series of non-perturbative contributions from instanton sectors. Let us neglect those instanton contributions for the moment. The perturbative low-energy effective coupling is [19]

$$\text{Im} \frac{\partial Z}{\partial a} = \frac{4p}{\pi} \ln \frac{|a|}{\Lambda} - \frac{p}{2\pi} \ln \frac{|m^2 - a^2|}{\Lambda^2}, \quad (16)$$

where we have introduced the dynamically generated scale Λ . When $|a| \gg |m|$, the coupling is given by the one-loop β function of the non-abelian $\text{SU}(2)$ super Yang–Mills theory with one flavor of quark. When $|m| \gg |a|$ the quark must be integrated out and the running with respect to a is given by the β function of the pure $\text{SU}(2)$ super Yang–Mills theory. Around the points $a = \pm m$, the low energy theory is an abelian gauge theory coupled to one light charged hypermultiplet, and the infrared divergence when $a = \pm m$ is governed by the usual infrared-free coupling of this theory. From (16) we deduce

$$\frac{\partial Z}{\partial a} = q + \frac{4ip}{\pi} \ln \frac{a}{\Lambda} - \frac{ip}{2\pi} \ln \frac{m^2 - a^2}{\Lambda^2}, \quad (17)$$

for some integer electric number q . This equation can be integrated by noting that $Z(a = 0, m) = 0$ because the monopole solution reduces to the vacuum when $a = 0$. The fermion number charge is then immediately derived from (13) and we recover (6). The ambiguity modulo $2i\pi$ in the logarithm is cleared up by requiring that $-p/2 \leq F \leq p/2$ in the conjugation symmetric limit. The fermion-induced fractional electric

charge of the monopole is also immediately obtained from (17) by using (12),

$$\frac{2Q_e}{g_{\text{YM,eff}}} = q + \frac{p}{2\pi} \text{Im} \ln \frac{m^2 - a^2}{a^8}, \quad (18)$$

in perfect agreement with (14) in the case $a_2 = 0$.

What about the instanton series? Eqs. (12) and (13) give a precise prescription from which the exact non-perturbative charges can in principle be calculated from the formulas of [13]. This is interesting, because to my knowledge the phenomenon of charge fractionization has never been studied beyond the semi-classical approximation. However, the exact charges are highly model-dependent and can be calculated only in a supersymmetric context. On the other hand, the results of the semi-classical approximation (6) or (18) entirely rely on the mathematical analysis of (3) and are universal and independent of supersymmetry.

The two-dimensional case with fermion number (7) can be treated similarly. The Dirac equation (3) occurs in the coupling of a charged chiral multiplet containing the fermion ψ with a twisted chiral multiplet containing A_μ and ϕ in a two-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. A review on this type of theory can be found in [20]. The parameter m is often called a twisted mass in this context. The central charge appears in the anticommutator

$$\{\bar{Q}_+, Q_-\} = 4Z. \quad (19)$$

For a soliton interpolating between two vacua ϕ_- and ϕ_+ , the classical central charge is expressed in terms of the tree-level twisted superpotential $W(\phi)$ and the fermion number,

$$Z_{\text{cl}} = i(W(\phi_+) - W(\phi_-)) + mF. \quad (20)$$

Quantum-mechanically, Z is a holomorphic function of the parameters and is expressed in terms of an effective superpotential W_{eff} deduced by integrating out the charged chiral multiplet. This amounts to a simple one-loop calculation because the multiplet appears only quadratically in the action. The result is

$$W_{\text{eff}} = \frac{1}{2\pi} (m + \phi) \ln \frac{m + \phi}{e\Lambda}. \quad (21)$$

The fermion number (7) is then immediately deduced from (13). In this case, there is no correction to the semi-classical formula for F as a function of

the vacuum expectation values ϕ_+ and ϕ_- of the scalar field ϕ . Yet, those expectation values are model-dependent and can pick up some non-perturbative terms.

3. Prospects

Apart from its simplicity, the most attractive feature of our approach is that it can a priori be generalized to string theory. String theory has solitonic states called D-branes which are very similar to magnetic monopoles. A detailed theory of charge fractionization for D-branes could then certainly be developed. It would be very interesting to work out the mathematical concepts that generalize the spectral asymmetry of the Dirac operator which is the central object in field theory. Our method suggests that string perturbation theory together with analyticity constraints could be used to calculate the charges.

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